Question 1

Let $f: R \to R$ be a differentiable function. Prove that there is a point $t \in (0,1)$ such that

$$f(1)-f(0)=\frac{\pi}{2}\sqrt{1-t^2}\,f'(t).$$

(Hint: Consider $g(x)=f(\sin x)$ for all $0 \le x \le \frac{\pi}{2}$)

(10 Marks)

Question 2

Let $a_n = \frac{2^3-1}{2^3+1}\cdot\frac{3^3-1}{3^3+1}\cdots\frac{n^3-1}{n^3+1}$. Find $\lim_{n\to\infty} a_n$.

(Hint: You may use the identity $a^2-a+1=(a-1)^2+(a-1)+1$)

(10 Marks)

Question 3

Find $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{k}{n}+\frac{1}{n}\right)$, where $\ln$ denotes the natural logarithm.

(10 Marks)

Question 4

Let A, B be $n \times n$ real matrices. If $AB = A + B$, show that $AB = BA$.

(10 Marks)

Question 5

Let V and W be finite dimensional vector spaces and $f: V \longrightarrow W$ be a linear transformation with kernel f is the set $Ker\,(f) = \{v | f(v) = 0\}$. Prove that f is one to one if and only if $Ker(f) = \{0\}$.

(10 Marks)

Question 6

A nonempty set S with a binary operation $*$ is called a semigroup if

i) $a, b \in S$ then $a * b \in S$

ii) $a*(b*c)=(a*b)*c$ for all $a,b,c \in S$.

Further, a nonempty set T with a ternary operation $\bullet$ is called a ternary semigroup if

i) $a,b,c \in T$ then $a \bullet b \bullet c \in T$

ii) $(a \bullet b \bullet c)\bullet d \bullet e = a \bullet (b \bullet c \bullet d) \bullet e = a \bullet b \bullet (c \bullet d \bullet e)$ for all $a,b,c,d,e \in T$.

Prove that S with this ternary operation $\bullet$ is a ternary semigroup.

(10 Marks)